

Large Deviations in the Spherical Model: The Rate Functions.

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Abstract. We study the spherical model of a ferromagnet in d -dimensional cubes Ω_n of volume $|\Omega_n| = n^d$ and investigate large deviations of the magnetization of various domains $D_k \subset \Omega_n$. We focus our attention on the low-temperature regime, $T < T_c$, and consider domains D_k of three types: $(d-1)$ -dimensional layers of width k , $(d-2)$ -dimensional rods, and Kadanoff blocks. In the case of layers the large-deviation probabilities decay exponentially with n^{d-2} , and we obtain an explicit expression for the corresponding rate function. When the layer width $k \ll n$, the large-deviation probabilities are virtually independent of k . In the case of rods the probabilities of large deviations exhibit similar exponential decay, but this time it is distorted by $\log n$ corrections. In the case of Kadanoff blocks of size k the large-deviation probabilities decay exponentially with k^{d-2} .

KEY WORDS: Critical phenomena; equivalence classes; correlation length; Kadanoff blocks.

1 Introduction.

Most models studied within the theory of critical phenomena describe the behaviour of various order parameters reasonably well. Qualitative predictions of mean-field and of short-range finite-dimensional models are very similar in this respect. The only significant discrepancy is in the values of critical exponents. The latter predicted by 2D and/or 3D models differ quite substantially from the set of mean-field critical exponents.

To find qualitative differences in the behaviour of various classes of models in the entire low-temperature region (below the critical point) one can look at the *probabilities of large deviations*:

$$\Pr [m_N \in [a, b]],$$

where m_N is the corresponding order parameter, N is the number of microscopic degrees of freedom in the system, and (if we are indeed talking about large deviations) the interval $[a, b]$ does not contain the equilibrium value, m^* , of the order

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parameter. Typically, as $N \rightarrow \infty$, large-deviation probabilities exhibit the following asymptotic behaviour:

$$-\frac{1}{N} \ln \Pr [m_N \in [a, b]] \sim \min_{x \in [a, b]} R(x),$$

where $R(x) \geq 0$ is the corresponding *rate function*.

Although, being the probabilities of (extremely) rare events, the above quantities may seem unimportant they have significant implications on, for instance, the existence/absence of such a mysterious phenomenon as the hysteresis loop. Large-deviation probabilities also describe the behaviour of thermodynamic systems when it is necessary to take into account some conservation laws. A frequently encountered example is the conservation of the number of particles. Properties of typical configurations realising appropriate large deviations in a system where the number of particles is not conserved are closely related to equilibrium properties of the same system with the conservation law.

For short-range finite-dimensional models the rate functions $R(x)$ are always convex functions with a continuous first derivative. On the contrary, in the low-temperature regions the rate functions of mean-field models are not convex. Therefore, in this respect the properties of mean-field models are unphysical.

For instance, the Hamiltonian of the Curie-Weiss model of a ferromagnet — the canonical mean-field model — is given by

$$H_N^{\text{cw}} = -\frac{J}{2N} \left(\sum_{j=1}^N s_j \right)^2 - h \sum_{j=1}^N s_j,$$

where $s_j = \pm 1$, for $j = 1, 2, \dots, N$. A straightforward calculation, see, e.g., [4], yields the following asymptotics

$$-\frac{1}{N} \ln \Pr \left[\frac{1}{N} \sum_{j=1}^N s_j = \frac{k}{N} \right] \sim F\left(\frac{k}{N}\right) - F\left(\frac{k^*}{N}\right),$$

for $k \in \{-N, -N+2, \dots, N-2, N\}$, where $\Pr[\cdot]$ is the Gibbs distribution corresponding to the Hamiltonian H_N^{cw} and the inverse temperature β ,

$$F(x) = -\frac{\beta J x^2}{2} - \beta h x + \frac{1-x}{2} \ln(1-x) + \frac{1+x}{2} \ln(1+x),$$

and k^* minimizes $F(\frac{k}{N})$. If $h = 0$ and $\beta J > 1$ the corresponding rate function

$$R_{\text{cw}}(x) = F(x) - \min_{y \in [-1, 1]} F(y) \tag{1}$$

is not convex, see Fig. 1. When $h \neq 0$ the function $R_{\text{cw}}(x)$ has exactly one global minimum. If $h (\neq 0)$ is sufficiently small $R_{\text{cw}}(x)$ also has a local minimum, which is often interpreted as a quasi-stationary state giving rise to the hysteresis phenomenon.

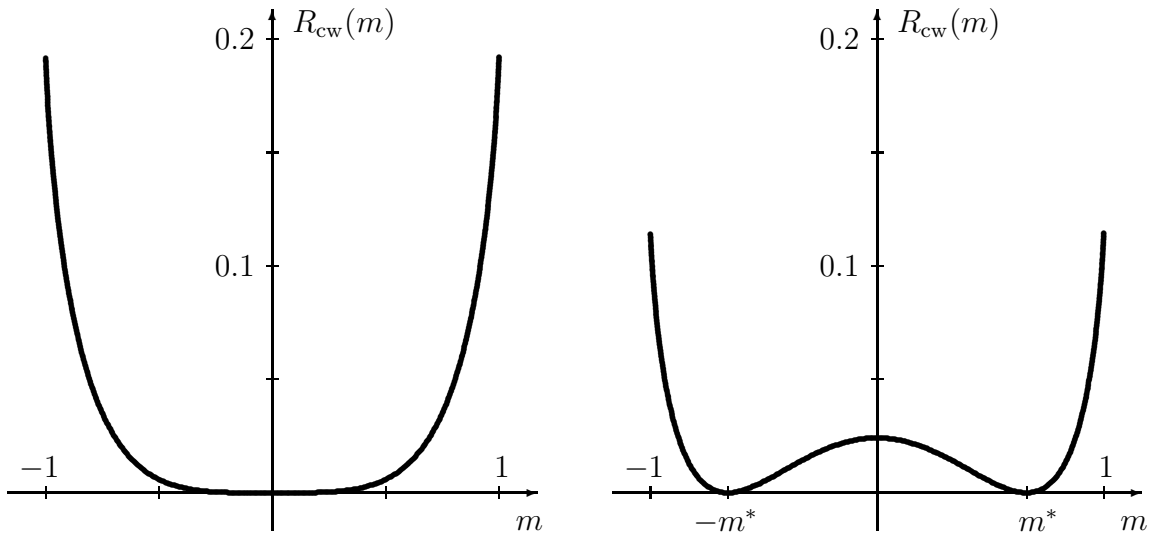


Figure 1: The rate functions for the magnetization of the Curie-Weiss model at $\beta = \beta_c = 1/J$ (left) and at $\beta = 1.2/J$ (right).

It is possible to consider a version of the Curie-Weiss model with continuous random variables s_j . In this case the corresponding rate function will differ from $R_{\text{cw}}(x)$ given by Eq. (1), but certain qualitative features of $R(x)$ (for instance, the non-convex shape in the low-temperature region) will remain unchanged. Thus, although the rate functions $R(x)$ are not universal quantities, some of their properties are identical within large classes of models.

Short-range finite-dimensional lattice models can be divided into two broad classes: continuous and discrete. A well-known exactly solvable representative of the former class is the spherical model of a ferromagnet defined in cubical domains Ω_n of a square lattice Z^d , see [1] and the definition in Section 2. A cube Ω_n contains $n^d \equiv N$ lattice sites. The order parameter here is the magnetization m_N and the rate function describing the large-deviation probabilities for m_N is given by $R_{\text{sph}}(x) = G(x) - G(0)$, where $G(x) = \max_{z \geq d} g(x, z)$ and

$$g(x, z) = -\beta J(1 - x^2)(z - d) + \frac{1}{2(2\pi)^d} \int_0^{2\pi} \dots \int_0^{2\pi} d\omega_1 \dots d\omega_d \ln \left(z - \sum_{\nu=1}^d \cos \omega_\nu \right),$$

see [6]. The function $R_{\text{sph}}(x)$ is always convex, see Fig. 2. In the low-temperature region $\beta > \beta_c$ this function vanishes on the entire interval $x \in [-m^*, m^*]$, where $m^* \equiv \sqrt{1 - \beta_c/\beta}$ is the spontaneous magnetization obtained by switching off a homogeneous external magnetic field $h \downarrow 0$. Switching on an arbitrarily weak magnetic field immediately leads to a strictly convex rate function with a positive second derivative and a unique minimum. Thus, when $h \neq 0$ the spherical model on a finite-dimensional lattice does not have quasi-stationary states, and hence it does not exhibit any hysteresis phenomena associated with such states.

Since the rate function $R_{\text{sph}}(x)$ is equal to zero on the entire interval $[-m^*, m^*]$ it is still necessary to find the first non-vanishing term in the large- n asymptotic expansion of $\ln \Pr [m_N \in [a, b]]$ when $[a, b] \cap [-m^*, m^*] \neq \emptyset$. It turns out that in the

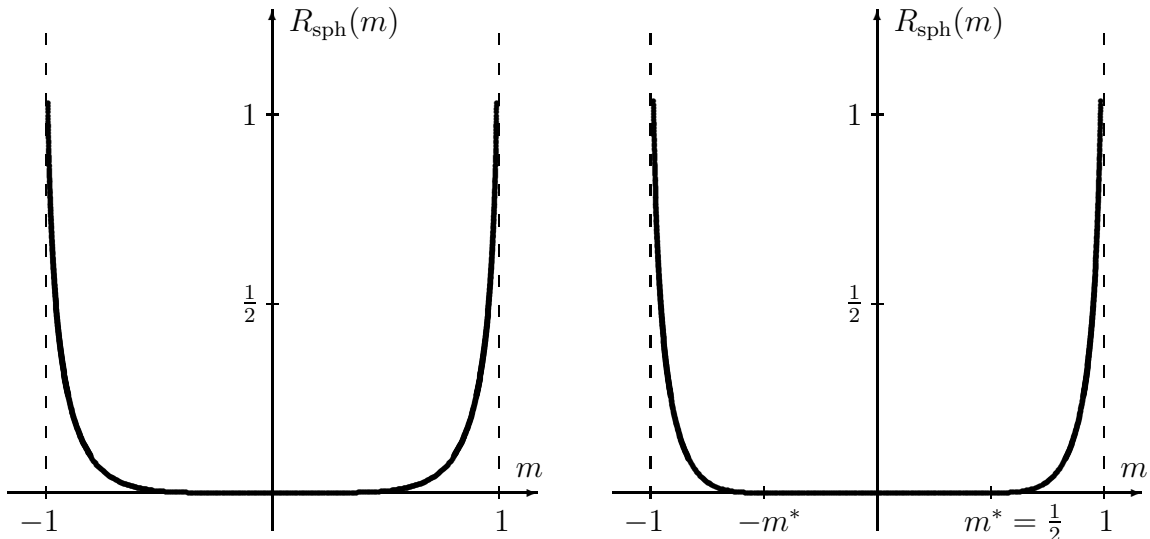


Figure 2: The rate functions for the magnetization of the spherical model at $\beta = \beta_c$ (left) and at $\beta = \frac{4}{3}\beta_c$ (right).

case of the spherical model with periodic boundary conditions this term is given by

$$\ln \Pr [m_N \in [a, b]] \sim -n^{d-2} \min_{x \in [a, b]} R_{\text{sph}}^{(2)}(x),$$

where

$$R_{\text{sph}}^{(2)}(x) = 2\pi^2 \beta J \left(1 - \frac{\beta_c}{\beta} - x^2 \right), \quad \text{for } x \in [-m^*, m^*],$$

see Fig. 3. Note that, unlike $R(x)$ — the rate function of the order n^d , lower-order rate functions (the order n^{d-2} here and the order n^{d-1} below) do not have to be convex functions with a continuous first derivative.

Among discrete models the 2D Ising model on a square $n \times n$ lattice is the most frequently studied example. The order parameter here is the magnetization m_N , $N \equiv n^2$. An explicit expression for the corresponding rate function, $R_{\text{Ising}}(m)$, is not known. However, it is known that $R_{\text{Ising}}(m)$ is always convex and, thus, it has the same shape as the analogous rate function within the spherical model shown in Fig. 2. In particular, for $\beta > \beta_c$ the function $R_{\text{Ising}}(m)$ vanishes on the entire interval $[-m^*, m^*]$, where $m^* \equiv [1 - \sinh^{-4}(2\beta J)]^{1/8}$ is the spontaneous magnetization. If $[a, b] \cap [-m^*, m^*] \neq \emptyset$, then the leading asymptotics of the large-deviation probabilities is given by

$$\ln \Pr [m_N \in [a, b]] \sim -n \min_{x \in [a, b]} R_{\text{Ising}}^{(1)}(x),$$

see [3, 7]. It is quite remarkable that despite $R_{\text{Ising}}(x)$ is still unknown, in many cases one can find an explicit expression for the rate function $R_{\text{Ising}}^{(1)}(x)$. For instance, it was shown in the paper [8] that in the case of the 2D Ising model with periodic boundary conditions this rate function (in the notations adopted to ours) is given by

$$R_{\text{Ising}}^{(1)}(x) = \beta w \times \begin{cases} \sqrt{m^* - |x|}, & \text{for } m_0 < |x| \leq m^*, \\ \sqrt{m^* - m_0}, & \text{for } |x| \leq m_0, \end{cases}$$

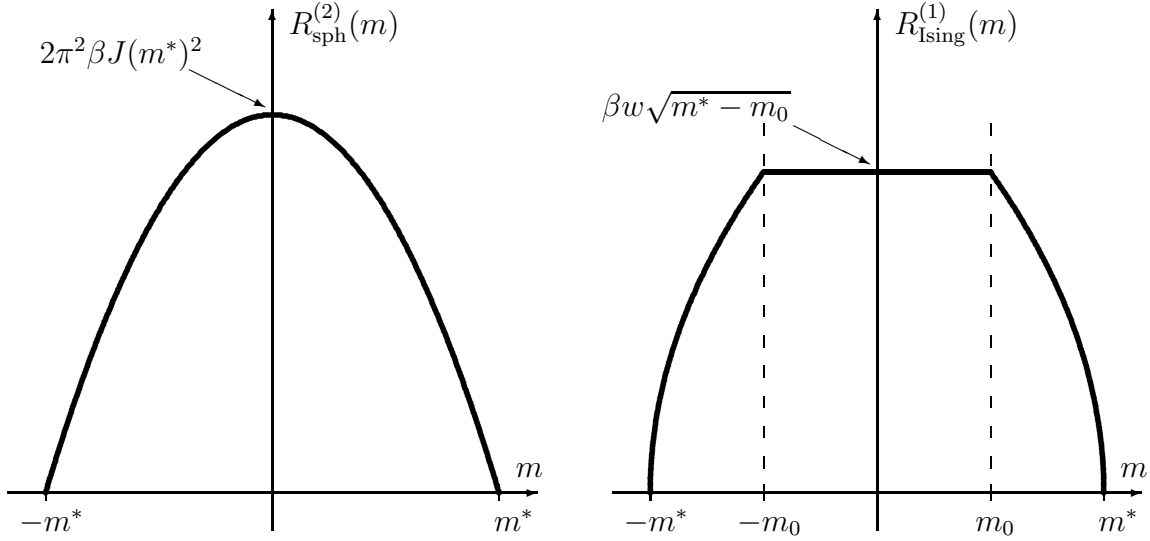


Figure 3: The rate functions $R_{\text{sph}}^{(2)}(m)$ and $R_{\text{Ising}}^{(1)}(m)$ for the magnetization of the 3D spherical and 2D Ising models in the low-temperature region $\beta > \beta_c$.

see Fig. 3, where w is the surface tension associated with the droplet boundary, and m_0 is the magnetization value at which the droplet shape changes from a rounded square to a ring taking advantage of the periodic boundary conditions.

The obvious qualitative differences in the shapes of $R_{\text{sph}}^{(2)}(m)$ and $R_{\text{Ising}}^{(1)}(m)$, see Fig. 3, are the slopes at $m = \pm m^*$ and the horizontal segment in $R_{\text{Ising}}^{(1)}(m)$ for $|m| \leq m_0$. Apparently, the absence of a threshold value m_0 in $R_{\text{sph}}^{(2)}(m)$ is due to a special structure of typical configuration in the spherical model with a fixed value of the magnetization. These configurations always take advantage of the periodic boundary conditions, and hence they are of the ring shape (as opposed to a round droplet shape). The absence of the horizontal segment $R_{\text{sph}}^{(2)}(m)$ is due to the diffuse nature of the boundary between the “+” and “−” phases. Increasing the amount of “+” or “−” phase inside the ring makes the interphase boundary steeper, and, hence, makes the set of corresponding configurations less probable.

Another well-known model, which is often attributed to the mean-field class, is the Ising model on a Cayley tree. The explicit expression for the rate function $R_{\text{Cayley}}(x)$ of the magnetization is not known, nevertheless, it is possible to compute the values of this function numerically. It was shown in the paper [2] that $R_{\text{Cayley}}(x)$ is always convex and vanishes at exactly one point where it has strictly positive second derivative. Without any doubt the same conclusion is valid for many other models on Cayley trees. If so, then one should conclude that ferromagnets on trees do not exhibit genuine critical behaviour, where single zero of the rate function either stretches to an interval, or splits into several zeroes for sufficiently low temperatures. Rather, models on trees are some kind of bundles of (non-critical) one-dimensional chains. Distributions of order parameters in tree models always concentrate at a single point (the rate-function zero) even for low temperatures and in the absence of symmetry breaking perturbations. Non-zero values of order parameters appear only as a result of explicit breaking of symmetry by a field applied to an abnormally

large number of boundary sites.

Thus, thermodynamic systems can be classified according to the qualitative behaviour of their large-deviation probabilities. One can put forward a hypothesis that several large classes can be described as follows. Rate functions, $R(x)$, of macroscopic observables (order parameters) in *non-critical* systems are always convex and vanish (reach zero value) at exactly one point. For sufficiently low temperatures (strong correlations) the rate functions of (long-range) *mean-field* models are not convex and vanish at a finite number of points. Rate functions of *discrete finite-dimensional* models exhibiting critical behaviour are convex and (when the temperature is low enough) they vanish on a certain interval $[-m^*, m^*]$. If $[a, b] \cap [-m^*, m^*] \neq \emptyset$, then

$$\ln \Pr [m_N \in [a, b]] \sim -n^{d-1} \min_{x \in [a, b]} R^{(1)}(x).$$

Rate functions $R(x)$ of *continuous finite-dimensional* models are similar to those of discrete models. However, if $[a, b] \cap [-m^*, m^*] \neq \emptyset$, then

$$\ln \Pr [m_N \in [a, b]] \sim -n^{d-2} \min_{x \in [a, b]} R^{(2)}(x).$$

The properties of typical configurations realising large-deviations are even more intriguing than the probabilities of these events. Hopefully those will be outlined in a consecutive publication.

Thus, large-deviation probabilities in short-range finite-dimensional systems are sensitive to the dimensionality of the lattice. In fact there also exists a dimension dependence of another kind. If we look at the magnetization of a subdomain D_k of the entire system Ω_n , then, as it turns out, the behaviour of the corresponding large-deviation probabilities is very sensitive to the shape or/and dimensionality of D_k . It is this feature that is the main focus of the present work.

The rest of the paper is organized as follows. Section 2 contains the definition of the spherical model and statements of the main results. Large deviations of the magnetization of various domains are studied in Section 3. Subsections 3.1 and 3.2 are devoted to the large-deviation probabilities for the magnetization of $(d-1)$ -dimensional layers and $(d-2)$ -dimensional rods, respectively. Subsection 3.3 contains derivation of analogous properties for the magnetization of Kadanoff blocks. The results of the paper are discussed in Section 4.

2 Definition of the Model and Main Results.

Let \mathbf{Z}^d be a d -dimensional square lattice with nodes $j \equiv (j_1, j_2, \dots, j_d)$, where $j_\nu \in \mathbf{Z}$ for $\nu = 1, 2, \dots, d$. Consider the sequence of cubes

$$\Omega_n = \{j \in \mathbf{Z}^d : 1 \leq j_\nu \leq n, \nu = 1, 2, \dots, d\}$$

where a random variable $\sigma_j \in \mathbf{R}^1$ (spin) is attached to each node $j \in \Omega_n$. The mutual dependence (interaction) of these spins is described by the Hamiltonian

$$H_n(\sigma) = -J \sum_{\langle i, j \rangle} \sigma_i \sigma_j, \quad J > 0; \quad (2)$$

where the summation runs over all pairs of nearest neighbours

$$\langle i; j \rangle \in \Omega_n \times \Omega_n, \quad \sum_{\nu=1}^d |i_\nu - j_\nu| = 1.$$

We assume that the periodic boundary conditions are imposed (for instance, the nodes $(1, j_2, \dots, j_d)$ and (n, j_2, \dots, j_d) are also nearest neighbours).

The joint probability distribution of the random variables $\{\sigma_j : j \in \Omega_n\}$ is specified by the density

$$p(\{\sigma_j : j \in \Omega_n\}) = \frac{1}{\Theta_n} \exp[-\beta H_n(\sigma)], \quad (3)$$

with respect to the “spherical” *a priori* measure

$$\mu_n(d\sigma) = \delta\left(\sum_{j \in \Omega_n} \sigma_j^2 - N\right) \prod_{j \in \Omega_n} d\sigma_j, \quad (4)$$

where Θ_n is the partition function (normalizing factor), $N \equiv n^d = |\Omega_n|$, $\delta(\cdot)$ is the Dirac delta function, and $\prod_{j \in \Omega_n} d\sigma_j$ is the Lebesgue measure on $(\mathbf{R}^N, \mathcal{B}(\mathbf{R}^N))$. Equations (2), (3), and (4) define the spherical model of a ferromagnet [1].

Consider the sequence of Kadanoff blocks

$$B_k \equiv \{j \in \Omega_n : 1 \leq j_\nu \leq k, \nu = 1, 2, \dots, d\},$$

where $k \equiv k(n)$ is a non-decreasing function of n (possibly a constant). The normalized total spin (magnetization) of a block B_k is given by

$$S_B = |B_k|^{-1} \sum_{j \in B_k} \sigma_j.$$

In the present paper we study the distribution densities

$$\phi_{n,B}(m) = \frac{1}{\Theta_n} \int_{-\infty}^{+\infty} \delta(S_B - m) \exp[-\beta H_n(\sigma)] \mu_n(d\sigma) \quad (5)$$

of the random variables S_B . In particular, we would like to investigate the asymptotic behaviour of $\phi_{n,B}(m)$ as n and $|B_k|$ tend to infinity.

To carry out the integration in Eq. (5) we need to know the spectral properties of the symmetric matrix \hat{C}_n associated with the Hamiltonian (2)

$$H_n(\sigma) = -J \sum_{i,j \in \Omega_n} C_{ij}^{(n)} \sigma_i \sigma_j.$$

For the periodic boundary conditions the eigenvalues and eigenvectors of \hat{C}_n are given by

$$\lambda_{j_1, \dots, j_d} = \sum_{\nu=1}^d \cos \frac{2\pi(j_\nu - 1)}{n}, \quad \mathbf{V}_{j_1, \dots, j_d} = \left\{ \prod_{\nu=1}^d v_{j_\nu, \ell_\nu} \right\}_{\ell \in \Omega_n}, \quad (6)$$

where

$$v_{j_\nu, \ell_\nu} = \frac{1}{\sqrt{n}} \left[\cos \frac{2\pi(j_\nu - 1)(\ell_\nu - 1)}{n} + \sin \frac{2\pi(j_\nu - 1)(\ell_\nu - 1)}{n} \right]. \quad (7)$$

The components v_{j_ν, ℓ_ν} can be written down in the following compact form

$$v_{j_\nu, \ell_\nu} = \sqrt{\frac{2}{n}} \cos \left[\frac{2\pi(j_\nu - 1)(\ell_\nu - 1)}{n} - \frac{\pi}{4} \right],$$

which we will use throughout the paper, although in all calculations we actually worked with Eq. (7).

No paper on the spherical model on a square lattice \mathbf{Z}^d can be written without a reference to the Watson function

$$W_d(z) \equiv \frac{1}{(2\pi)^d} \int_{-\pi}^{+\pi} \dots \int_{-\pi}^{+\pi} \frac{d\omega_1 \dots d\omega_d}{z - \sum_{\nu=1}^d \cos \omega_\nu}. \quad (8)$$

The integral of this function also appears quite frequently in various expressions and certainly deserves a special symbol

$$L_d(z) \equiv \frac{1}{(2\pi)^d} \int_{-\pi}^{+\pi} \dots \int_{-\pi}^{+\pi} d\omega_1 \dots d\omega_d \ln \left(z - \sum_{\nu=1}^d \cos \omega_\nu \right). \quad (9)$$

The main results of the present paper can be stated as follows. Consider the Kadanoff blocks B_k and also the following subdomains of the cubes Ω_n :

$$\begin{aligned} L_k &= \{j \in \Omega_n : 1 \leq j_1 \leq k\}, \\ R_k &= \{j \in \Omega_n : 1 \leq j_\nu \leq k, \nu = 1, 2\}, \end{aligned}$$

which we call layers and rods. Let $1 \ll k \ll n$, then the large- n asymptotics of the large-deviation probabilities for the magnetization of layers, rods, and blocks are given by

$$\begin{aligned} \log \Pr [m_L \in [a, b]] &\sim -n^{d-2} \min_{x \in [a, b]} R_L^{(2)}(x), \\ \log \Pr [m_R \in [a, b]] &\sim -\frac{n^{d-2}}{\log n} \min_{x \in [a, b]} R_R^{(2)}(x), \\ \log \Pr [m_B \in [a, b]] &\sim -k^{d-2} \min_{x \in [a, b]} R_B^{(2)}(x), \end{aligned}$$

where explicit expressions for the rate functions are given by Eqs. (23), (31), and (36), respectively.

3 Magnetization of Kadanoff blocks.

To find the distribution density of the magnetization of a Kadanoff block B , see Eq. (5), we have to calculate the following integral

$$\Theta_{n,B} = \int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} \prod_{j \in \Omega_n} d\sigma_j \delta \left(\sum_{j \in \Omega_n} \sigma_j^2 - N \right) \delta \left(\frac{1}{|B|} \sum_{j \in B} \sigma_j - m \right) e^{-\beta H_n(\sigma)}. \quad (10)$$

The integration over σ_j , $j \in \Omega_n$ is carried out using the standard technique after Berlin and Kac, see [1]. Namely, first a new set of integration variables y_l , $l \in \Omega_n$ is introduced via $\sigma_j = \sum_{l \in \Omega_n} V_{j,l} y_l$, where $\{V_{j,l}\}_{j \in \Omega_n} \equiv \mathbf{V}_l$, $l \in \Omega_n$ are the eigenvectors of the matrix \hat{C}_n associated with the Hamiltonian $H_n(\sigma)$, see Eqs. (6) and (7). Next, the delta functions are replaced by their integral representations

$$\delta(a) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{i\tau a} d\tau,$$

the integration order is exchanged, and the integration over the variables y_l , $l \in \Omega_n$ is performed. Note that the integration order can be switched only if the quadratic form in the argument of the exponential function is negatively defined. This can be achieved by a shift of the integration contour for τ . One obtains then

$$\Theta_{n,B} = \frac{1}{4\pi^2 i} \int_{-i\infty+\tau_0}^{i\infty+\tau_0} d\tau e^{\tau N} \int_{-\infty}^{\infty} dx e^{-ixm} \prod_{j \in \Omega_n} \sqrt{\frac{\pi}{\tau - \beta J \lambda_j}} \exp \left[-\frac{x^2 \gamma_j^2}{4(\tau - \beta J \lambda_j)} \right], \quad (11)$$

where $\gamma_j \equiv |B|^{-1} \sum_{l \in B} V_{j,l}$, and τ_0 is the shift of integration contour mentioned above. For the periodic boundary conditions a straightforward calculation yields the following expression for the coefficients γ_j , $j \equiv (j_1, j_2, \dots, j_d)$:

$$\gamma_j = |B|^{-1} (2/n)^{d/2} \prod_{\nu=1}^d \frac{\sin[\pi k(j_\nu - 1)/n]}{\sin[\pi(j_\nu - 1)/n]} \cos \left[\frac{\pi(k-1)(j_\nu - 1)}{n} - \frac{\pi}{4} \right]. \quad (12)$$

On integrating over the variable x in Eq. (11) and on introducing a new integration variable z via $\tau = \beta J z$ one arrives at

$$\Theta_{n,B} = \frac{\beta J}{2\pi i} \left(\frac{\pi}{\beta J} \right)^{(N-1)/2} \int_{-i\infty+z_0}^{i\infty+z_0} \frac{dz}{\sqrt{\Sigma(z)(z - \lambda_{(1,\dots,1)})}} \exp [N\beta J \Phi_n(z, m)], \quad (13)$$

where the following notations have been introduced

$$\Sigma(z) = \sum_{j \in \Omega_n} \frac{\gamma_j^2}{z - \lambda_j}, \quad (14)$$

$$\Phi_n(z, m) = z - \frac{1}{2\beta J N} \sum_{j \in \Omega_n \setminus (1,\dots,1)} \log(z - \lambda_j) - \frac{1}{N} \frac{m^2}{\Sigma(z)}. \quad (15)$$

The large- n asymptotic expansion for the remaining integral over z can be derived using the saddle-point method. This is quite straightforward when $\beta < \beta_c$, where $\beta_c = W_d(d)/(2J)$ is the inverse critical temperature of the spherical model, see [1], and $W_d(d)$ is the Watson function (8) at $z = d$. However, if $|B| = o(n^d)$, some extra efforts are required in the low-temperature region $\beta > \beta_c$. Difficulties arise because the sequence of saddle points $\{z_n^*\}$ (that is, the sequence of minimum points of the function $\Phi_n(z, m)$ on (d, ∞)) converges to $z = d$, as $n \rightarrow \infty$. That is, the sequence of saddle points z_n^* approaches the maximal singularity, ξ_n , of $\Phi_n(z, m)$: $z_n^* - \xi_n \downarrow 0$, as $n \rightarrow \infty$. Therefore, the standard version of the saddle-point method can not be

applied, and the integral (13) needs a special investigation in the low-temperature region. What we actually have to do is to introduce a new integration variable ζ via $z = d + n^{-\rho}\zeta$, where the exponent ρ must be chosen in such a way that the sequence of function $\tilde{\Phi}_n(\zeta) \equiv \Phi_n(d + n^{-\rho}\zeta, m)$ has a “conventional” saddle-point landscape in the limit $n \rightarrow \infty$.

The main goal of the present paper is investigation of large deviation probabilities for the magnetization of cubic Kadanoff blocks B . However, it turns out that the large-deviation probabilities for the magnetization of layers

$$L_k = \{j \in \Omega_n : 1 \leq j_1 \leq k\}$$

and rods

$$R_k = \{j \in \Omega_n : 1 \leq j_\nu \leq k, \nu = 1, 2\}$$

differ qualitatively from those for magnetization of cubic blocks B . Therefore, we begin from the simplest (as it happens to be) case of layers L , after that we consider large deviations for rods R , and finally we turn our attention to the case of Kadanoff blocks B .

3.1 Large deviations of the magnetization of layers L .

If the boundary conditions are periodic in all dimensions and

$$L_k \equiv \{j \in \Omega_n : 1 \leq j_1 \leq k\}, \quad (16)$$

then a straightforward calculation yields the following expression for the coefficients $\gamma_j \equiv |L_k|^{-1} \sum_{l \in L_k} V_{j,l}$ (see Eq. (11)):

$$\gamma_j = \frac{\sqrt{2} n^{\frac{d}{2}-1}}{|L_k|} \frac{\sin[\pi k(j_1 - 1)/n]}{\sin[\pi(j_1 - 1)/n]} \cos \left[\frac{\pi(k-1)(j_1 - 1)}{n} - \frac{\pi}{4} \right] \prod_{\nu=2}^d \delta_{1,j_\nu}. \quad (17)$$

Hence, in the case of layers L_k the multiple sum in the expression for $\Sigma(z)$ (see Eq. (14)) reduces to a single one and can be calculated exactly using the method described in [5]. One obtains

$$|L_k| \Sigma(z) = \frac{1}{z-d} \left[1 - \frac{2}{k(x_2 - x_1)} \frac{(1 - x_1^k)(x_2^n - x_2^k)}{(x_2^n - 1)} \right], \quad (18)$$

where $x_{1,2} = 1 + z - d \mp \sqrt{(z-d)(2+z-d)}$.

Now we are going to locate the maximal singularity of the function $\Phi_n(z, m)$ given by Eq. (15). That will give us a hint how one should rescale the integration variable z in order to preserve the saddle-point profile of $\Phi_n(z, m)$ in the limit $n \rightarrow \infty$. Let $k = \alpha n^\gamma$, where $\gamma \in (0; 1)$ and $\alpha > 0$. It is obvious from Eq. (14) that singularities of the function $\Sigma(z)$ — simple poles — are at the points

$$z \in \{\lambda_{(j,1,\dots,1)} : \gamma_{(j,1,\dots,1)} \neq 0, j = 1, \dots, n\}.$$

To locate zeroes of this function we rescale the variable z according to $z = d + \zeta n^{-\rho}$, where ζ is a new independent variable. The rescaling yields as $n \rightarrow \infty$:

$$|L_k|\Sigma(d + \zeta n^{-\rho}) = \begin{cases} n^\rho/\zeta + O(n^{3/2\rho}/k), & \text{for } 0 \leq \rho < 2\gamma, \zeta > 0 \\ \frac{n^{2\gamma}}{\zeta} \left[1 - \frac{1 - \exp(-\alpha\sqrt{2\zeta})}{\alpha\sqrt{2\zeta}} \right] + O(1), & \text{for } \rho = 2\gamma, \zeta > 0 \\ \frac{kn^{\rho/2}}{\sqrt{2\zeta}} + O(k^2), & \text{for } 2\gamma \leq \rho < 2, \zeta > 0 \\ kn/h(\zeta) + O(k^2), & \text{for } \rho = 2, \end{cases} \quad (19)$$

where

$$h(\zeta) = \begin{cases} \sqrt{2\zeta} \tanh \sqrt{\zeta/2}, & \text{for } \zeta \geq 0, \\ -\sqrt{-2\zeta} \tan \sqrt{-\zeta/2}, & \text{for } \zeta < 0. \end{cases}$$

Hence, the function $\Sigma(z)$ vanishes at the points

$$z_f = d - \frac{1}{2}\pi^2(1 + 2f)^2 n^{-2} + O(kn^{-3}), \quad f = 0, 1, 2, \dots \quad (20)$$

Obviously, the integrand in Eq. (13) is an analytic function for $z > \lambda_{(1,\dots,1)}$. The function $\Sigma(z)$ has a singularity at the point $z = \lambda_{(1,\dots,1)}$. However, for the periodic boundary conditions the largest eigenvalue of the interaction matrix \hat{C}_n is non-degenerate, and the singularity of $\Sigma(z)$ is cancelled by the multiplier $z - \lambda_{(1,\dots,1)}$. Thus, the point $z = \lambda_{(1,\dots,1)}$ is a removable singularity of the integrand. According to Eq. (20) the function $\Sigma(z)$ has a simple zero on the interval $(\lambda_{(2,1,\dots,1)}; \lambda_{(1,\dots,1)})$. Consequently, the maximal singularity of the integrand in Eq. (13) is at the point $z = s_n \equiv d - \frac{1}{2}\pi^2 n^{-2} + O(kn^{-3})$. The obtained asymptotic expansion for the location of the maximal singularity suggests that the change of integration variable $z = d + \zeta n^{-2}$ might preserve the saddle-point profile of the function $\Phi_n(z, m)$ in the limit $n \rightarrow \infty$.

Note now that for any $\zeta > -\frac{1}{2}\pi^2$ we have

$$\Phi_n(d + \zeta n^{-2}, m) = d - \frac{L_d(d)}{2\beta J} + n^{-2} \left[\zeta \left(1 - \frac{\beta_c}{\beta} \right) - m^2 h(\zeta) \right] + O(kn^{-3}),$$

as $n \rightarrow \infty$, where $L_d(d)$ is given by Eq. (9). Hence, the function $\Phi_n(z, m)$ attains its minimum on the interval $(s_n; \infty)$ at the point $z_n^* = d + n^{-2}\zeta^*(m) + O(kn^{-3})$, where $\zeta^*(m)$ is the maximal solution of

$$m^2 h'(\zeta) = 1 - \frac{\beta_c}{\beta}.$$

Obviously $\zeta^*(m) > -\frac{1}{2}\pi^2$ unless $m = 0$. The point $z = z_n^*$ is the saddle point of the integrand in Eq. (13).

In the scale $z = d + \zeta n^{-2}$ the saddle point $\zeta^*(m)$ does not approach the maximal singularity $s_n \equiv d - \frac{1}{2}\pi^2 n^{-2} + O(kn^{-3})$ of the function $\Phi_n(d + \zeta n^{-2}, m)$. Hence the function $\tilde{\Phi}_n(\zeta) \equiv \Phi_n(d + \zeta n^{-2}, m)$ has a conventional saddle-point landscape in the limit $n \rightarrow \infty$. Therefore in the scale $z = d + \zeta n^{-2}$ one can find the asymptotic

expansion for the integral Eq. (13) using the standard saddle-point method and obtain

$$\Theta_{n,L} = \exp \left\{ -Nf(\beta) + n^{d-2}\beta J \left[\zeta^*(m) \left(1 - \frac{\beta_c}{\beta} \right) - m^2 h(\zeta^*(m)) \right] + O(kn^{d-3}) \right\},$$

where

$$f(\beta) = \frac{1}{2} \log(\beta J / \pi) - \beta J d + \frac{1}{2} L_d(d) \quad (21)$$

is the limiting free energy per spin of the spherical model for $\beta > \beta_c$. The partition function of the spherical model with the periodic boundary conditions in the canonical ensemble is given by

$$\Theta_n = \exp \{ -Nf(\beta) + O(\log n) \}. \quad (22)$$

Thus, we obtain the following asymptotics for the distribution density of the magnetization of the layer L , see Eq. (5):

$$\phi_{n,L}(m) = \exp \left\{ -n^{d-2} R_L^{(2)}(m) + O(kn^{d-3}) \right\},$$

as $n \rightarrow \infty$, where the rate function is given by

$$R_L^{(2)}(m) = -\beta J \left[\zeta^*(m) \left(1 - \frac{\beta_c}{\beta} \right) - m^2 h(\zeta^*(m)) \right]. \quad (23)$$

The function $R_L^{(2)}(m)$ has a cusp at $m = 0$ and vanishes at the points $m = \pm \sqrt{1 - \beta_c/\beta}$ — the equilibrium values of the magnetization within the spherical model with periodic boundary conditions, see Fig. 3. The nature of the cusp at $m = 0$ is obvious. A magnetization value m can be achieved by deforming either equilibrium $\langle + \rangle$ -phase or $\langle - \rangle$ -phase. For negative (positive) values of m a deformation of minus (plus) phase is the easiest way to achieve the desired value of magnetization. As $m \uparrow 0$ the required deformations become more and more costly, because we are getting further and further away from the equilibrium value $-m^*$. However, if we keep increasing the value of m deformations become less costly as soon as we cross the point $m = 0$, because we begin to approach the equilibrium value m^* .

3.2 Large deviations of the magnetization of rods R .

In this subsection we investigate large deviations of the magnetization of $(d-2)$ -dimensional domains (rods)

$$R_k = \{j \in \Omega_n : 1 \leq j_\nu \leq k, \nu = 1, 2\}.$$

As in the previous subsection the large-deviation probabilities are determined by the relative location of the two largest eigenvalues of the interaction matrix and the maximal zero of the function $\Sigma(z)$. However in the case of rods R_k the distances between the eigenvalues and zeroes are qualitatively different from what we found

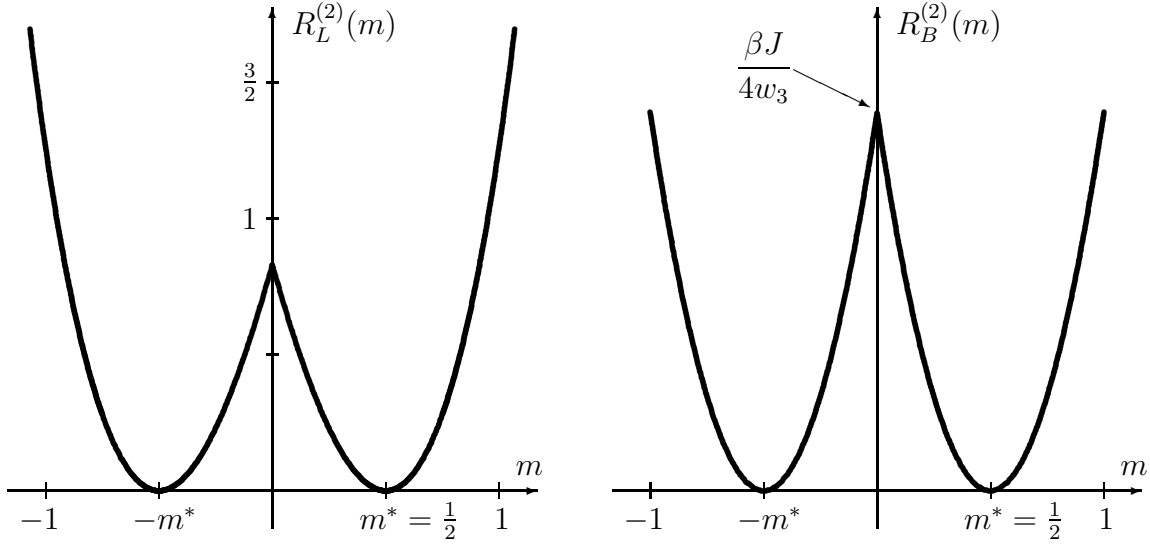


Figure 4: The rate functions for the magnetisation of the layers L (left) and the Kadanoff blocks B (right) at $\beta = \frac{4}{3}\beta_c$.

in the case of layers L_k . As a result, the probabilities of large deviations are also qualitatively different in these two cases.

In the case of rods R_k the coefficients $\gamma_j \equiv |R_k|^{-1} \sum_{l \in R_k} V_{j,l}$ are given by

$$\gamma_j = \frac{2}{|R_k| n^{d/2-2}} \prod_{\nu=1,2} \frac{\sin[\pi k(j_\nu - 1)/n]}{\sin[\pi(j_\nu - 1)/n]} \cos \left[\frac{\pi(k-1)(j_\nu - 1)}{n} - \frac{\pi}{4} \right] \prod_{\kappa=3}^d \delta_{1,j_\kappa}. \quad (24)$$

Hence the multiple sum in the expression for $\Sigma(z)$ (see Eq. (14)) reduces to the double one

$$\Sigma(z) = \frac{n^{d-4}}{|R_k|^2} \sum_{j_1, j_2=1}^n \frac{\sin^2[\pi k(j_1 - 1)/n]}{\sin^2[\pi(j_1 - 1)/n]} \frac{\sin^2[\pi k(j_2 - 1)/n]}{\sin^2[\pi(j_2 - 1)/n]} \times \frac{1}{z - d + 2 - \cos[2\pi(j_1 - 1)/n] - \cos[2\pi(j_2 - 1)/n]}.$$

The function $\Sigma(z)$ has simple poles at the points $z = d$ and $z = d - 1 + \cos(2\pi/n)$ and a simple zero s_n^* somewhere between the poles. Our objective now is to find an expression for $\Sigma(z)$ convenient enough for locating s_n^* .

In the case of $(d-1)$ -dimensional layers the expression for $\Sigma(z)$ contained only a single sum, which we managed to calculate exactly. For $(d-2)$ -dimensional rods we do not have a luxury of working with an exact closed-form expression for $\Sigma(z)$, nevertheless, fortunately, simplifications arise for a different reason. As is often the case for double sums of this type $\Sigma(d + \zeta n^{-2}) = O(n^{d-2} k^4 \log n)$, for $\zeta > 0$. Under the same rescaling the term corresponding to $j_1 = j_2 = 0$ (the contribution from the maximal eigenvalue of the interaction matrix) is given by $n^{d-2} k^4 / \zeta$. Denote now $\Sigma'(z)$ the same sum as in Eq. (14) but with the contribution from the maximal eigenvalue omitted. Below we will argue that the main asymptotics of $\Sigma'(d + \zeta n^{-2})$ is given by $n^{d-2} k^4 \log n \sigma(\zeta)$, where $\sigma(\zeta)$ is a positive analytic function for $\zeta > -2\pi^2$.

Therefore, in order to find s_n^* , we have to solve the equation $\sigma(\zeta) \log n = -\zeta^{-1}$. Since the function $\sigma(\zeta)$ is regular at $\zeta = 0$, the solution is given by

$$\zeta^* = -\frac{1}{\sigma(0) \log n} + O\left(\frac{1}{\log^2 n}\right),$$

and hence

$$s_n^* \sim d - \frac{1}{\sigma(0)n^2 \log n}, \quad \text{as } n \rightarrow \infty.$$

Let $k = \alpha n^\gamma$, where $\alpha > 0$ and $\gamma \in (0; 1)$. For $z = d + \zeta n^{-\rho}$, where $\zeta > 0$ and $0 \leq \rho < 2$, one has as $n \rightarrow \infty$

$$\Sigma(d + \zeta n^{-\rho}) = n^{d-2} I_2(\zeta n^{-\rho}) + O\left[\exp\left(-\sqrt{2\zeta} n^{1-\rho/2}\right)\right],$$

where

$$I_m(x) = \frac{1}{(2\pi)^m} \int_0^{2\pi} \dots \int_0^{2\pi} d\omega_1 \dots d\omega_m \prod_{\nu=1}^m \frac{1 - \cos(k\omega_\nu)}{1 - \cos \omega_\nu} \frac{1}{x + m - \sum_{\kappa=1}^m \cos \omega_\kappa}. \quad (25)$$

For $z = d + \zeta n^{-2}$ one obtains

$$\Sigma(d + \zeta n^{-2}) = n^{d-2} \left\{ I_2(\zeta n^{-2}) + k^4 \left[\delta_1(\zeta) + \delta_2(\zeta) + O(n^{-(1-\gamma)/2}) \right] \right\},$$

where

$$\delta_1(\zeta) = \sum_{j=-\infty}^{\infty} \frac{2}{\sqrt{2(\zeta + 2\pi^2 j^2)}} \frac{1}{\exp \sqrt{2(\zeta + 2\pi^2 j^2)} - 1},$$

and

$$\delta_2(\zeta) = \frac{1}{\pi} \int_0^\infty \frac{2}{\sqrt{2(\zeta + 2\omega^2)}} \frac{d\omega}{\exp \sqrt{2(\zeta + 2\omega^2)} - 1}. \quad (26)$$

Note that $\lambda_{(1,\dots,1)} - \lambda_{(2,1,\dots,1)} \sim 2\pi^2 n^{-2}$, hence, to locate s_n^* we have to consider the scale $z = d + \zeta n^{-2}$ or even a finer one. In any rougher scale the poles and zeroes of $\Sigma(z)$ merge as $n \rightarrow \infty$, and the function $\Phi_n(d + \zeta n^{-\rho}, m)$ does not have a required saddle-point landscape.

Isolating the singularity of the integral $I_2(\zeta n^{-2})$ one can write it down in the following form

$$I_2(\zeta n^{-2}) = \frac{k^4}{(2\pi)^2} \iint_{\omega_1^2 + \omega_2^2 \leq c} \frac{d\omega_1 d\omega_2}{\zeta n^{-2(1-\gamma)} + \frac{1}{2}(\omega_1^2 + \omega_2^2)} + k^4 r_n^{(1)}(\zeta),$$

where the function $r_n^{(1)}(\zeta)$ (the regular part of the integral) is analytic in the unit circle $\{\zeta \in C : |\zeta| < 1\}$. As $n \rightarrow \infty$ the sequence $r_n^{(1)}(\zeta)$ converges to some limiting function uniformly over the unit circle. Calculating the double integral one obtains

$$I_2(\zeta n^{-2}) = \frac{k^4}{2\pi} \left[\log \frac{n^{2(1-\gamma)}}{\zeta} + \log \left(\frac{1}{2} c^2 + \zeta n^{-2(1-\gamma)} \right) \right] + k^4 r_n^{(1)}(\zeta). \quad (27)$$

Separating the singularities of $\delta_1(\zeta)$ at $\zeta = 0$ one obtains

$$\delta_1(\zeta) = \frac{1}{\zeta} - \frac{1}{\sqrt{2\zeta}} + r^{(2)}(\zeta), \quad (28)$$

where $r^{(2)}(\zeta)$ is analytic for $\text{Re } \zeta > -2\pi^2$. Analogous separation of singularities at $\zeta = 0$ yields

$$\delta_2(\zeta) = \frac{2}{\pi} \int_0^\infty \frac{d\omega}{2\zeta + \omega^2} - \frac{1}{\pi} \int_0^c \frac{d\omega}{\sqrt{2\zeta + \omega^2}} + r_3(\zeta) = \frac{1}{\sqrt{2\zeta}} + \frac{\log \zeta}{2\pi} + \tilde{r}_3(\zeta), \quad (29)$$

where the functions $r_3(\zeta)$ and $\tilde{r}_3(\zeta)$ are analytic in the unit circle. Summarizing Eqs. (27), (28), and (29) one arrives at

$$\Sigma(d + \zeta n^{-2}) = n^{d-2} k^4 \left[\frac{1-\gamma}{\pi} \log n + \zeta^{-1} + \tilde{r}_n(\zeta) \right], \quad (30)$$

where $\tilde{r}_n(\zeta)$ are analytic and bounded uniformly over n in the unit circle. Hence, the maximal zero of the function $\Sigma(z)$ is at the point

$$s_n^* = d - \frac{\pi}{1-\gamma} \frac{1}{n^2 \log n} + O[(n \log n)^{-2}].$$

The expression for s_n^* suggests that the change of variable $z = d + \zeta(n^2 \log n)^{-1}$ is likely to convert Eq. (13) into an integral convenient for application of the saddle-point method. Substituting $z = d + \zeta(n^2 \log n)^{-1}$ in the expression for $\Phi_n(z)$ we see that this is indeed the case. One obtains

$$\begin{aligned} \Phi_n \left(d + \frac{\zeta}{n^2 \log n} \right) = \\ n^d \beta J \left[d - \frac{L(d)}{2\beta J} \right] + \frac{n^{d-2} \beta J}{\log n} \left[\zeta \left(1 - \frac{\beta_c}{\beta} \right) - \frac{\pi m^2}{1 - \gamma + \pi \zeta^{-1}} \right] + O \left(\frac{n^{d-2}}{\log^2 n} \right). \end{aligned}$$

Hence, the sequence of relevant saddle points is given by

$$z_n^* = d + \frac{\zeta^*}{n^2 \log n} + O \left(\frac{1}{n^2 \log^2 n} \right),$$

where

$$\zeta^* = \frac{\pi}{1-\gamma} \left(\frac{|m|}{\sqrt{1 - \beta_c/\beta}} - 1 \right).$$

Evaluating the integral in Eq. (13) using the saddle-point method one obtains

$$\Theta_{N,R} = \exp \left[N f(\beta) - \frac{n^{d-2}}{\log n} \frac{\pi \beta J}{1-\gamma} \left(|m| - \sqrt{1 - \beta_c/\beta} \right)^2 + O \left(\frac{n^{d-2}}{\log^2 n} \right) \right].$$

Taking into account Eq. (22) one arrives at

$$\phi_{n,R}(m) = \exp \left[-\frac{n^{d-2}}{\log n} R_R^{(2)}(m) + O \left(\frac{n^{d-2}}{\log^2 n} \right) \right],$$

where

$$R_R^{(2)}(m) = \frac{\pi \beta J}{1-\gamma} \left(|m| - \sqrt{1 - \beta_c/\beta} \right)^2. \quad (31)$$

3.3 Large deviations in Kadanoff blocks.

In this section we investigate the probabilities of large deviations for the total spin of the Kadanoff blocks

$$B_k = \{j \in \Omega_n : 1 \leq j_\nu \leq k, \nu = 1, \dots, d\}.$$

Qualitative behaviour of the corresponding large-deviation probabilities is the same for any $d \geq 3$. Therefore, to avoid unnecessary technical complications we consider only the case $d = 3$. The coefficients $\gamma_j \equiv |B_k|^{-1} \sum_{l \in B_k} V_{j,l}$ are given by

$$\gamma_j = \frac{2^{3/2}}{|B_k|} \prod_{\nu=1,2,3} \frac{\sin[\pi k(j_\nu - 1)/n]}{\sin[\pi(j_\nu - 1)/n]} \cos \left[\frac{\pi(k-1)(j_\nu - 1)}{n} - \frac{\pi}{4} \right], \quad (32)$$

and

$$\Sigma(z) = n^{-3} \sum_{j_1, j_2, j_3=1}^n \prod_{\nu=1}^3 \frac{\sin^2[\pi k(j_\nu - 1)/n]}{\sin^2[\pi(j_\nu - 1)/n]} \frac{1}{z - \sum_{\kappa=1,2,3} \cos[2\pi(j_\kappa - 1)/n]}.$$

The function $\Sigma(z)$ has simple poles at the points $z = d$ and $z = d - 1 + \cos(2\pi/n)$ and a simple zero s_n^* in the interval $(d - 1 + \cos(2\pi/n); d)$. Our goal now is to obtain an expression for $\Sigma(z)$ convenient enough for locating s_n^* .

Let $k = \alpha n^\gamma$, with $\alpha > 0$ and $\gamma \in (0; 1)$. For $z = d + \zeta n^{-2}$, $\zeta > 0$, one has as $n \rightarrow \infty$

$$\Sigma(d + \zeta n^{-2}) = I_3(\zeta n^{-2}) + \frac{k^6}{n} \left[\delta_1(\zeta) + \delta_2(\zeta) + \delta_3(\zeta) + O(n^{-(1-\gamma)/2}) \right],$$

where $I_3(x)$ is given by Eq. (25),

$$\begin{aligned} \delta_1(\zeta) &= \sum_{j,l=-\infty}^{\infty} \frac{2}{\sqrt{2\zeta + 4\pi^2(j^2 + l^2)}} \frac{1}{\exp \sqrt{2\zeta + 4\pi^2(j^2 + l^2)} - 1}, \\ \delta_2(\zeta) &= \frac{1}{\pi} \sum_{j=-\infty}^{\infty} \int_0^\infty \frac{2}{\sqrt{2\zeta + 4\pi^2 j^2 + \omega^2}} \frac{d\omega}{\exp \sqrt{2\zeta + 4\pi^2 j^2 + \omega^2} - 1}, \end{aligned} \quad (33)$$

and

$$\delta_3(\zeta) = \frac{1}{\pi^2} \iint_0^\infty d\omega_1 d\omega_2 \frac{2}{\sqrt{2\zeta + \omega_1^2 + \omega_2^2}} \frac{1}{\exp \sqrt{2\zeta + \omega_1^2 + \omega_2^2} - 1}. \quad (34)$$

The large- n asymptotics of $I_3(\zeta n^{-\rho})$ for $\rho > 2\gamma$ is given by

$$I_3(\zeta n^{-\rho}) \sim k^5 w_3,$$

where

$$w_3 = \frac{2}{\pi^3} \iiint_{-\infty}^{\infty} d\omega_1 d\omega_2 d\omega_3 \prod_{\nu=1}^3 \frac{1 - \cos \omega_\nu}{\omega_\nu^2} \frac{1}{\sum_{\kappa=1}^3 \omega_\kappa^2}. \quad (35)$$

Separating the singularities of $\delta_\nu(\zeta)$, $\nu = 1, 2, 3$ at $\zeta = 0$ (cf., Eqs. (27)–(29)) one obtains

$$\sum_{\nu=1,2,3} \delta_\nu(\zeta) = \zeta^{-1} + r(\zeta),$$

where the function $r(\zeta)$ is analytic in the unit ball $\{\zeta \in C : |\zeta| < 1\}$. Hence, the maximal zero of the function $\Sigma(z)$ is at the point

$$z = s_n \sim d - \frac{k}{n^3 w_3}.$$

The location of the maximal zero gives us a hint that the asymptotic expansion of the integral (13) can be found using the saddle-point method after a prior rescaling of the integration variable z via $z = d + \zeta n^{-3+\gamma}$. The rescaling yields

$$\Phi_n(d + \zeta n^{-3+\gamma}) = d - \frac{L(d)}{2\beta J} + kn^{-3} \left[\zeta \left(1 - \frac{\beta_c}{\beta} \right) - \frac{m^2}{w_3 + \zeta^{-1}} \right] + o(kn^{-3}).$$

Hence, for $\beta > \beta_c$ the sequence of relevant saddle points is given by

$$z_n^* = d + \zeta^* kn^{-3} + o(kn^{-3}),$$

where

$$\zeta^* = \frac{1}{w_3} \left(\frac{|m|}{\sqrt{1 - \beta_c/\beta}} - 1 \right).$$

Evaluation of the integral (13) using the saddle-point method yields

$$\Theta_{N,B} = \exp \left[Nf(\beta) - k \frac{\beta J}{w_3} \left(|m| - \sqrt{1 - \beta_c/\beta} \right)^2 + o(k) \right].$$

Taking into account Eq. (22), one obtains the following asymptotic formula for the probability density of magnetization

$$\phi_{n,B} = \exp \left[-k R_B^{(2)}(m) + o(k) \right],$$

where

$$R_B^{(2)}(m) = \frac{\beta J}{w_3} \left(|m| - \sqrt{1 - \beta_c/\beta} \right)^2. \quad (36)$$

4 Discussion and concluding remarks.

In the present paper we have investigated large-deviation probabilities for the magnetization of various domains $D_k (\subset \Omega_n)$ within the spherical model of a ferromagnet defined in cubes Ω_n . We have shown that these probabilities are very sensitive to the shape and/or dimensionality of the domains D_k .

An appealing feature of the large-deviation theory is that often the behavior of large-deviation probabilities admits a simple intuitively clear explanation, see, e.g., [8]. This is also the case for the results obtained in the present paper. In Section

3.1 we have shown that in the case of layers L_k the large-deviation probabilities decay exponentially with n^{d-2} , which is similar to the asymptotics of the analogous probabilities for the magnetization of the entire cube Ω_n , see [6]. The reason for such similarity becomes clear when we investigate the properties of typical configurations realizing these deviations. It turns out that the easiest way to obtain a desired value of magnetization in the layer L_k is to deform the configuration of random variables in the entire cube Ω_n . That is, a large-deviation of the magnetization in a layer L_k leads to a large-deviation of the magnetization in the entire cube Ω_n . To put it another way, a substantial deformation of the configuration in a layer L_k spreads over the entire cube Ω_n .

In the case of rods R_k the large-deviation probabilities are modified by $\log n$ corrections, namely, they decay exponentially with $n^{d-2}/\log n$. Investigation of typical configurations realizing these large deviations shows that a substantial deformation of the configuration in a rod R_k does not spread over the entire cube Ω_n , but it spreads over a domain with the linear size of the order $n/\log n$. Note that if k satisfies the bound $\sqrt{nk'} \ll k \ll n$, then the number of random variables in a rod R_k is greater than that in a layer $L_{k'}$. Nevertheless, large deviations of the magnetization in the rod are more likely than large deviations in the layer simply because of a more compact arrangement of the random variables in the rod.

In the case of Kadanoff blocks B_k the large-deviation probabilities decay exponentially with k^{d-2} . Even a substantial deformation of the configuration in a block B_k does not spread very far from the block. Essentially it remains localized in a domain of the linear size $O(k)$, that is, of the same extent as the size of the Kadanoff block itself. The number of random variables in a block B_k can be greater than the number of random variables in a rod $R_{k'}$ (or a layer $L_{k''}$), nevertheless, a large deviation of the magnetization of the block is much more likely than a large deviation in the rod or the layer. Accordingly, a uniform deformation of the configuration in a layer $L_{k''}$ or a rod $R_{k'}$ spreads much further over the cube Ω_n than a deformation of a Kadanoff block B_k .

In order to find the properties of typical configurations realizing large deviations in layers, rods, and blocks one has to continue the calculation of the present paper one step further and derive the conditional distributions of random variables σ_j given a desired large deviation. These calculations will be published elsewhere.

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